Mathathon 2019
Round 3

Maths and Physics Club, IIT Bombay

2^{nd} October, 2019

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Freshie/Senior

For $n \in \mathbb{N}$, let $\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n.

1. For positive odd integer n, let $f(n)$ denote the number of matrices A satisfying the following conditions:

   (a) A is $n \times n$.

   (b) Each row and column contains each of $1,\ 2,\ \ldots,\ n$ exactly once in some order.

   (c) $A^T = A$.

   Prove that $f(n) \geq \dfrac{n!(n-1)!}{\varphi(n)}$.

2. Let n be a positive odd integer greater than 2, and consider a regular $n-gon$ $\mathcal{G}$ in the plane centered at the origin. Let a subpolygon $\mathcal{G}'$ be a polygon with at least 3 vertices whose vertex set is a subset of that of $\mathcal{G}$. We say that $\mathcal{G}'$ is *well-centered* if its centroid is the origin. Also, $\mathcal{G}'$ is decomposable if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

3. For integer $n \geq 4$, find the minimal integer $f(n)$, such that for any positive integer m, in any subset with $f(n)$ elements of the set $\{m,\ m+1,\ \ldots,\ m+n-1\}$ there are at least 3 mutually prime elements.

4. For any positive integer, let $N = \varphi(1) = \varphi(2) + \cdots \varphi(n)$. Show that there exists a sequence

   $$a_1,\ a_2,\ \ldots,\ a_N$$

   containing exactly $\varphi(k)$ instances of k for all positive integers $k \leq n$ such that

   $$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_N a_1} = 1.$$

5. Give a positive integer $n > 1000$, add the residues of 2^n modulo each of the numbers $1,\ 2,\ \ldots,\ n$. Prove that this sum is greater than $2n$.

6. Let p be an odd prime number. Find the number of subsets of $\{1,\ 2,\ \ldots,\ p\}$ with the sum of elements divisible by p.

7. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 0$ and $a_{4n} = a_{2n} + 1$, $a_{4n+1} = a_{2n} - 1$, $a_{4n+2} = a_{2n+1} - 1$, $a_{4n+3} = a_{2n+1} + 1$. Find the maximum and minimum values of a_n for $n = 1, 2, \ldots, 1996$ and the values of n at which they are attained. How many terms a_n for $n = 1, 2, \ldots, 1996$ are 0?