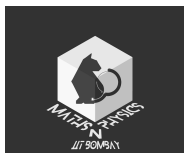


Mathathon 2019

Round 3



Maths and Physics Club, IIT Bombay

2nd October, 2019

Name:
E-mail:
Freshie/Senior

For $n \in \mathbb{N}$, let $\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n .

1. For positive odd integer n , let $f(n)$ denote the number of matrices A satisfying the following conditions:

- A is $n \times n$.
- Each row and column contains each of $1, 2, \dots, n$ exactly once in some order.
- $A^T = A$.

Prove that $f(n) \geq \frac{n!(n-1)!}{\varphi(n)}$.

2. Let n be a positive odd integer greater than 2, and consider a regular n -gon \mathcal{G} in the plane centered at the origin. Let a subpolygon \mathcal{G}' be a polygon with at least 3 vertices whose vertex set is a subset of that of \mathcal{G} . We say that \mathcal{G}' is *well-centered* if its centroid is the origin. Also, \mathcal{G}' is decomposable if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.
3. For integer $n \geq 4$, find the minimal integer $f(n)$, such that for any positive integer m , in any subset with $f(n)$ elements of the set $\{m, m+1, \dots, m+n-1\}$ there are at least 3 mutually prime elements.
4. For any positive integer, let $N = \varphi(1) + \varphi(2) + \dots + \varphi(n)$. Show that there exists a sequence

$$a_1, a_2, \dots, a_N$$

containing exactly $\varphi(k)$ instances of k for all positive integers $k \leq n$ such that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_N a_1} = 1.$$

5. Give a positive integer $n > 1000$, add the residues of 2^n modulo each of the numbers $1, 2, \dots, n$. Prove that this sum is greater than $2n$.
6. Let p be an odd prime number. Find the number of subsets of $\{1, 2, \dots, p\}$ with the sum of elements divisible by p .
7. The sequence a_1, a_2, a_3, \dots is defined by $a_1 = 0$ and $a_{4n} = a_{2n} + 1$, $a_{4n+1} = a_{2n} - 1$, $a_{4n+2} = a_{2n+1} - 1$, $a_{4n+3} = a_{2n+1} + 1$. Find the maximum and minimum values of a_n for $n = 1, 2, \dots, 1996$ and the values of n at which they are attained. How many terms a_n for $n = 1, 2, \dots, 1996$ are 0?